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Primordial Gravitational Waves Enhancement

Maria G. Romania[†] and N. C. Tsamis[‡]

*Institute of Theoretical & Computational Physics, and
Department of Physics, University of Crete
GR-710 03 Heraklion, HELLAS.*

R. P. Woodard^{*}

*Department of Physics, University of Florida
Gainesville, FL 32611, UNITED STATES.*

ABSTRACT

We reconsider the enhancement of primordial gravitational waves that arises from a quantum gravitational model of inflation. A distinctive feature of this model is that the end of inflation witnesses a brief phase during which the Hubble parameter oscillates in sign, changing the usual Hubble friction to anti-friction. An earlier analysis of this model was based on numerically evolving the graviton mode functions after guessing their initial conditions near the end of inflation. The current study is based on an equation which directly evolves the normalized square of the magnitude. We are also able to make a very reliable estimate for the initial condition using a rapidly converging expansion for the sub-horizon regime. Results are obtained for the energy density per logarithmic wave number as a fraction of the critical density. These results exhibit how the enhanced signal depends upon the number of oscillatory periods; they also show the resonant effects associated with particular wave numbers.

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[†] e-mail: romania@physics.uoc.gr

[‡] e-mail: tsamis@physics.uoc.gr

^{*} e-mail: woodard@phys.ufl.edu

1 Introduction

The case for a phase of accelerated expansion (*inflation*) during the very early universe is strong. One reason is that we can observe widely separated parts of the early universe which seem to be in thermal equilibrium with one another [1]. If one assumes the universe never underwent a period of inflation, there would not have been time for this thermal equilibrium to be established by causal processes. Without primordial inflation the number of causally distinct regions in our past light-cone at the time of recombination is over 10^3 , and it would be 10^9 at the time of nucleosynthesis.

There is no strong indication for what caused primordial inflation. A natural mechanism for inflation can be found within gravitation – which, after all, plays the dominant role in shaping cosmological evolution – by supposing that the bare cosmological constant Λ is not unnaturally small but rather large and positive.¹ Because Λ is constant in *space*, no special initial condition is needed to start inflation. We also dispense with the need to employ a new, otherwise undetected scalar field. However, Λ is constant in *time* as well, and classical physics can offer no natural mechanism for stopping inflation once it has begun [2]. Quantum physics can: accelerated expansion continually rips virtual infrared gravitons out of the vacuum [3] and these gravitons attract one another, thereby slowing inflation [4]. This is a very weak effect for $G\Lambda \ll 1$, but a cumulative one, so inflation lasts a long time for no other reason than that gravity is a weak interaction [4].

This screening mechanism may be clear enough on the perturbative level but it has two frustrating features. The first is that, because inflationary particle production is a 1-loop effect, the gravitational response to it is delayed until 2-loop order. The second frustration is that the 2-loop effect becomes unreliable just when it starts to get interesting. The effective coupling constant is $G\Lambda H t$ and higher loops are insignificant as long as it is small. But *all* loops become comparable when $G\Lambda H t$ becomes of order one, and the correct conclusion then is that perturbation theory breaks down. The breakdown occurs not because any single graviton-graviton interaction gets strong but rather because there are so many of them.

We believe it may be possible to derive a non-perturbative resummation technique by extending the stochastic method which Starobinsky devised for

¹Here “large” means a Λ induced by a matter scale which can be as high as 10^{18} GeV . Then, the value of the dimensionless coupling constant can be as high as $G\Lambda \sim 10^{-4}$ rather than the putative value of 10^{-122} .

the same purpose in scalar potential models [5, 6, 7]. However, generalizing this technique to gravity is a difficult problem [8]. This paper is part of an effort which is based on the idea of *guessing* the most cosmologically significant part of the effective field equations of quantum gravity. While there is no chance of guessing the full effective field equations, it might be possible to guess just enough to correctly describe the evolution of the scale factor $a(t)$ for a homogeneous and isotropic geometry, using what we know from perturbation theory about how the back-reaction effect scales. Such simple cosmological models were recently constructed [9, 10] and are reviewed in Section 2. Some basics of linearized gravitons needed in this work are the subject of Section 3. The possibility of getting an enhancement of high frequency gravity waves within this class of models was first investigated in [11]. In this paper we re-investigate this possibility using more accurate calculational techniques and we present our results in a way more appropriate for the needs of gravitational wave experiments. In Section 4 we review the enhancement mechanism while in Section 5 we derive an equation for the square of the magnitude of the mode functions and describe our improved evolution strategy. Our results are presented in Section 6. Their physical consequences and our concluding remarks comprise Sections 7 and 8.

2 The Cosmological Model

In a previous paper [9] we proposed a phenomenological model which can provide evolution beyond perturbation theory. In one sentence, we constructed an *effective* conserved stress-energy tensor $T_{\mu\nu}[g]$ which modifies the gravitational equations of motion:²

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}[g] . \quad (1)$$

and which, we hope, contains the most cosmologically significant part of the full effective quantum gravitational equations.

What form to guess for $T_{\mu\nu}[g]$ was motivated by what we seek to do, and by what we know from perturbation theory. We seek to describe cosmology, which implies homogeneous and isotropic geometries. When specialized to

²Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has spacelike signature and our curvature tensor equals: $R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\nu\beta,\mu} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\beta} - (\mu \leftrightarrow \nu)$. The initial Hubble parameter is $3H_0^2 \equiv \Lambda$.

such a geometry the full effective stress tensor must take the perfect fluid form and we lose nothing by assuming that generally:

$$T_{\mu\nu}[g] = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} . \quad (2)$$

The relation between $p[g]$, $\rho[g]$ and $u_\mu[g]$ is heavily constrained by stress-energy conservation, but it is possible to specify one function for free. It turns out to be computationally simplest to take this free function to be the pressure [9]. We further require the pressure to be an ordinary function of some non-local scalar which grows like the number of e-foldings when specialized to de Sitter. If the pressure is to grow the way we know it does from perturbation theory [7], and to eventually end inflation, then a simple choice has the form [9]:

$$p[g](x) = \Lambda^2 f[-G\Lambda X](x) \quad , \quad X \equiv \frac{1}{\square} R \quad , \quad (3)$$

where the function f grows without bound and satisfies:

$$f[-G\Lambda X] = -G\Lambda X + O[(G\Lambda)^2] \quad , \quad (4)$$

and where the scalar d'Alembertian:

$$\square \equiv \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu) \quad , \quad (5)$$

is defined with retarded boundary conditions. The induced energy density $\rho[g]$ and 4-velocity $u_\mu[g]$ are determined, up to their initial value data, from stress-energy conservation:

$$D^\mu T_{\mu\nu} = 0 \quad . \quad (6)$$

The 4-velocity was chosen to be timelike and normalized:

$$g^{\mu\nu} u_\mu u_\nu = -1 \quad \implies \quad u^\mu u_{\mu;\nu} = 0 \quad . \quad (7)$$

The homogeneous and isotropic evolution ³ of this model – using a combination of numerical and analytical methods – revealed the following basic

³The line element in co-moving coordinates is $ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}$. In terms of the scale factor a , the Hubble parameter equals $H(t) = \dot{a} a^{-1}$ and the deceleration parameter equals $q(t) = -a \ddot{a} \dot{a}^{-2} = -1 - \dot{H} H^{-2} \equiv -1 + \epsilon(t)$.

features: ⁴

- After the onset and during the era of inflation, the source $X(t)$ grows while the curvature scalar $R(t)$ and Hubble parameter $H(t)$ decrease.
- Inflationary evolution dominates roughly until we reach a critical point X_{cr} defined by:

$$1 - 8\pi G\Lambda f[-G\Lambda X_{cr}] \equiv 0 . \quad (8)$$

- The epoch of inflation ends close to but before the universe evolves to the critical time. This is most directly seen from the deceleration parameter since initially $q(t=0) = -1$ while at criticality $q(t=t_{cr}) = +\frac{1}{2}$.
- Oscillations in $R(t)$ become significant as we approach the end of inflation; they are centered around $R=0$, their frequency equals:

$$\omega = G\Lambda H_0 \sqrt{72\pi f'_{cr}} , \quad (9)$$

where H_0 is the constant inflationary Hubble parameter, and their envelope is linearly falling with time.

- During the oscillations era, although there is net expansion, the oscillations of $H(t)$ take it to small negative values for short time intervals – a feature conducive to rapid reheating; those of $\dot{H}(t)$ take it to positive values for about half the time; and, those of $a(t)$ are centered around a linear increase with time.

A novel feature of this class of models is the existence of an oscillatory regime of short duration which commences towards the very end of the inflationary era. During this period $\dot{H}(t)$ is positive about half the time, which represents a violation of the weak energy condition. Such a violation cannot occur in classically stable theories [12] but it can be driven by quantum effects of the type we seek to model without endangering stability [13].

3 Linearized Gravitons

In terms of the full metric field $g_{ij}(x)$, the fluctuating graviton field $h_{ij}^{TT}(x)$ is defined as:

$$g_{ij}(t, \mathbf{x}) = a^2(t) \left[\delta_{ij} + \sqrt{32\pi G} h_{ij}^{TT}(t, \mathbf{x}) \right] . \quad (10)$$

⁴In [9], our analytical results were obtained for any function f satisfying (4) and growing without bound, our numerical results for the choice: $f(x) = \exp(x) - 1$.

The free field expansion of the graviton field is:

$$h_{ij}^{TT}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \left\{ u(t, k) e^{i\mathbf{k} \cdot \mathbf{x}} \epsilon_{ij}(\mathbf{k}, \lambda) \alpha(\mathbf{k}, \lambda) + (c.c.) \right\} , \quad (11)$$

where $(c.c.)$ denotes complex conjugation, the polarizations $\epsilon_{ij}(\mathbf{k}, \lambda)$ and operators $\alpha(\mathbf{k}, \lambda)$ obey:

$$\epsilon_{ij}(\mathbf{k}, \lambda) \epsilon_{ij}^*(\mathbf{k}, \lambda') = \delta_{\lambda\lambda'} \quad , \quad \epsilon_{ii}(\mathbf{k}, \lambda) = k_i \epsilon_{ij}(\mathbf{k}, \lambda) = 0 \quad , \quad (12)$$

$$\left[\alpha(\mathbf{k}, \lambda) , \alpha^\dagger(\mathbf{k}', \lambda') \right] = \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \quad , \quad (13)$$

and the mode functions $u(t, k)$ satisfy:

$$\ddot{u}(t, k) + 3H(t) \dot{u}(t, k) + \frac{k^2}{a^2(t)} u(t, k) = 0 \quad , \quad (14)$$

with the Wronskian associated with the two solutions of (14) equaling:

$$u \dot{u}^* - \dot{u} u^* = i a^{-3} \quad . \quad (15)$$

We shall be interested in the energy $E(t, k)$ at time t of a mode with wavenumber k . The simplest way to derive this is to exploit the fact that the physical degrees of freedom of linearized gravitons have the same dynamics with those of a massless, minimally coupled scalar field $\varphi(x)$.⁵ The scalar field Lagrangian density is:

$$\mathcal{L}(x) = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = \frac{1}{2} a^3(t) \dot{\varphi}^2 - \frac{1}{2} \nabla \varphi \cdot \nabla \varphi \quad . \quad (16)$$

The Lagrangian diagonalizes in momentum space:

$$L(t) = \int d^3x \mathcal{L}(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2} a^3(t) |\dot{\tilde{\varphi}}(t, \mathbf{k})|^2 - \frac{1}{2} a(t) k^2 |\tilde{\varphi}(t, \mathbf{k})|^2 \right\} \quad (17)$$

so that any mode with wavenumber \mathbf{k} evolves independently as a harmonic oscillator $q(t)$ with time-dependent mass $m(t) \equiv a^3(t)$ and angular frequency $\omega(t) \equiv k a^{-1}(t)$:

$$q(t) = u(t, k) A + u^*(t, k) A^\dagger \quad , \quad [A, A^\dagger] = 1 \quad , \quad (18)$$

$$E_{SHO}(t) = \frac{1}{2} a^3(t) \dot{q}^2(t) + \frac{1}{2} a(t) k^2 q(t) \quad . \quad (19)$$

⁵The analogous computation within the linearized graviton theory should only make an $O(1)$ change to the result.

At any instant t the minimum energy is $E_{\min}(t, k) = \frac{1}{2}k a^{-1}(t)$. However since both the mass and angular frequency are time-dependent, the state with minimum energy at one time instant is not the state with minimum energy at another time instant; there is particle production as time evolves. The Bunch-Davies vacuum $|\Omega\rangle$ is the minimum energy state in the distant past and the expectation value of the energy operator (19) in its presence equals:

$$\langle\Omega| E(t, k) |\Omega\rangle = \frac{1}{2} a^3(t) |\dot{u}(t, k)|^2 + \frac{1}{2} k^2 a(t) |u(t, k)|^2 . \quad (20)$$

A fair measure of the excess energy $\Delta E(t, k)$ acquired during time evolution in any one wavenumber is obtained by subtracting the instantaneous minimum energy from (20):

$$\begin{aligned} \Delta E(t, k) &\equiv \langle\Omega| E(t, k) |\Omega\rangle - E_{\min}(t, k) \\ &= \frac{1}{2} a^3(t) |\dot{u}(t, k)|^2 + \frac{1}{2} k^2 a(t) |u(t, k)|^2 - \frac{k}{2a} . \end{aligned} \quad (21)$$

4 The Enhancement Mechanism

The oscillatory phase is a very distinctive feature of these models and in [11] we investigated the possibility of gravitational wave enhancement due to its presence. There are two very plausible physical arguments that convinced us this is a worthwhile inquiry:

- During the oscillations era the Hubble parameter $H(t)$ changes sign and this, in turn, changes the sign of the “friction” term $3H\dot{u}$ in the evolution equation (14) obeyed by the mode functions $u(t, k)$. For $H(t) > 0$ this term tends to reduce $|\dot{u}(t, k)|$ whereas it tends to increase $|\dot{u}(t, k)|$ when $H(t) < 0$. What happens to the magnitude $|u(t, k)|$ depends upon where $u(t, k)$ is in its own oscillations when $H(t)$ changes sign but the change from “friction” to “anti-friction” can clearly strengthen the amplitude in some cases.
- The oscillations era is characterized by the frequency ω given by (9). Gravitational waves of frequency close to ω can resonate and their amplitude can increase.

The first effort to evolve (14) through the oscillatory phase was done in [11]. As expected, it is the *near-horizon* modes that experience enhancement: the natural time scale of their $u(t, k)$ is close to the inverse of the oscillatory

frequency ω and we get a significant resonance response.⁶ When converted to current frequencies, the main conclusion of [11] is the enhancement of gravitational waves with frequencies somewhat less than 10^{10} Hz. In obtaining these results, however, certain assumptions were necessary since we do not possess exact forms for the two linearly independent solutions of (14) during the oscillatory regime. Nor do we know which linear combination of these two solutions is the actual mode function as we do not know the linear combination coefficients.⁷ The latter are determined by knowledge of the initial conditions at criticality. Because the post-inflationary scale factor effectively describes an overall linear expansion on which the oscillations are superimposed [9], in [11] we solved (14) for a linearly expanding $a(t)$ – which can be done exactly – and then numerically superimposed the effect of the oscillations. We also had to make an “educated guess” regarding the initial conditions at criticality.

In re-visiting the subject, we have developed a method – to be described in the next Section – which is considerably more accurate and, therefore, leads to robust conclusions.

5 The Evolution Strategy

- *The Variable $M(t, k)$*

We wish to derive an equation for the quantity $M(t, k)$:

$$M(t, k) \equiv u(t, k) u^*(t, k) = |u(t, k)|^2 , \quad (22)$$

because it is directly related to the tensor power spectrum $\Delta_h^2(t, k)$:

$$\begin{aligned} \Delta_h^2(t, k) &\equiv 32\pi G \frac{k^3}{2\pi^2} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \Omega | h_{ij}^{TT}(t, \mathbf{x}) h_{ij}^{TT}(t, \mathbf{0}) | \Omega \rangle \\ &= \frac{16\pi G}{\pi} k^3 M(t, k) . \end{aligned} \quad (23)$$

From the definition of M it follows that:

$$\dot{M} = \dot{u} u^* + u \dot{u}^* , \quad (24)$$

$$\ddot{M} = \ddot{u} u^* + 2\dot{u} \dot{u}^* + u \ddot{u}^* . \quad (25)$$

⁶Here and throughout, *super-, sub-, near- horizon* is with respect to the modes k_{cr} whose first horizon crossing occurred at criticality, when the transition from the inflationary to the oscillating era occurred: $k_{\text{cr}} = H(t_{\text{cr}}) a(t_{\text{cr}})$.

⁷The actual mode function is the coefficient of the annihilation operator in the free field expansion of the graviton.

By using the fact that \ddot{u} satisfies (14) we conclude:

$$\ddot{M} + 3H\dot{M} + \frac{2k^2}{a^2}M = 2\dot{u}\dot{u}^* . \quad (26)$$

By subtracting the square of (15) from that of (24), we can express the right hand side of (26) in terms of M and \dot{M} :

$$\dot{u}\dot{u}^* = \frac{1}{4M} \left[\dot{M}^2 + \frac{1}{a^6} \right] , \quad (27)$$

and obtain the desired equation:

$$\ddot{M} + 3H\dot{M} + \frac{2k^2}{a^2}M = \frac{1}{2M} \left[\dot{M}^2 + \frac{1}{a^6} \right] . \quad (28)$$

The goal is to find $M(t, k)$ such that (28) is obeyed. An exact solution is beyond our abilities but we can divide the full time evolution range into separate intervals and obtain reliable approximate expressions for $M(t, k)$ within each of these.

- *The Evolution of $M(t, k)$: Inflation*

- During the inflationary era, it makes sense to adopt a scheme that works accurately for any kind of mode and, at the same time, avoids numerical evolution for as long as possible. A method that seems optimal is the development of an asymptotic series expansion for $M(t, k)$ in powers of $H^2 a^2 \div k^2$:

$$M(t, k) = \frac{1}{2k a^2} \left\{ 1 + \alpha(t) \left(\frac{Ha}{k} \right)^2 + \beta(t) \left(\frac{Ha}{k} \right)^4 + \dots \right\} . \quad (29)$$

Substituting the above in (28) allows us to determine the leading coefficients $\alpha(t)$, $\beta(t)$ of the series. The final form for the asymptotic expansion of $M(t, k)$ becomes:

$$M(t, k) = \frac{1}{2k a^2} \left\{ 1 + \left(1 - \frac{\epsilon}{2} \right) \left(\frac{Ha}{k} \right)^2 + \left[\frac{9}{4} \epsilon - \frac{21}{8} \epsilon^2 + \frac{3}{4} \epsilon^3 + \right. \right. \\ \left. \left. + \left(\frac{7}{4} - \frac{3\epsilon}{4} \right) \frac{\dot{\epsilon}}{H} + \frac{\ddot{\epsilon}}{8H^2} \right] \left(\frac{Ha}{k} \right)^4 + \dots \right\} . \quad (30)$$

As long as ϵ does not get large, the series (30) converges rapidly and we can use it to evolve all the way to within, say, 2 e-foldings before first horizon

crossing.⁸ We shall, therefore, adopt this method and evolve very accurately: (i) any *sub-horizon* mode all the way to criticality, (ii) any *near-horizon* mode until, say, 2 e-foldings before criticality, and (iii) any *super-horizon* mode until, say, 2 e-foldings before first horizon crossing. Afterwards, in all cases equation (28) is evolved numerically.

The important cosmological parameters in the inflationary era are [9, 11]:

$$a(t) = a_{\text{cr}} e^{-N} , \quad (31)$$

$$H(t) \simeq \frac{1}{3} \omega \sqrt{4N + \frac{4}{3}} , \quad (32)$$

$$\dot{H}(t) \simeq -\frac{2H^2}{4N + \frac{4}{3}} , \quad (33)$$

$$\epsilon(t) \simeq \frac{2}{4N + \frac{4}{3}} , \quad (34)$$

where N is the number of e-foldings before criticality. The initial conditions used in the numerical analysis are those inherited from (30) at the appropriate time.

• *The Evolution of $M(t, k)$: Oscillations*

During the oscillatory era the important cosmological parameters are [9, 11]:

$$a(t) = a_{\text{cr}} C_2 \left[C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi) \right] , \quad (35)$$

$$H(t) = \frac{\omega \left[1 - \sqrt{2} \sin(\omega \Delta t + \phi) \right]}{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)} , \quad (36)$$

$$\dot{H}(t) = -H^2(t) - \frac{\omega^2 \sqrt{2} \cos(\omega \Delta t + \phi)}{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)} , \quad (37)$$

$$\epsilon(t) = 1 + \frac{\sqrt{2} \cos(\omega \Delta t + \phi) \left[C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi) \right]}{\left[1 - \sqrt{2} \sin(\omega \Delta t + \phi) \right]^2} , \quad (38)$$

where $\Delta t \equiv t - t_{\text{cr}}$ measures time with respect to criticality. In this regime we analyze equation (28) numerically. The parameters (ϕ, C_1, C_2) in (35-38) are chosen to match the outcome from the inflationary epoch (31-34) at

⁸The error is in ignoring terms proportional to $\left(\frac{Ha}{k}\right)^6$ and higher. Even when we reach 2 e-foldings before first horizon crossing that is very small: $\left(\frac{Ha}{k} = e^{-2}\right)^6 = e^{-12}$.

criticality, where $N = 0$ and $\Delta t = 0$:

$$\phi = \arcsin\left(\frac{\sqrt{2} - \sqrt{2970}}{56}\right) \approx -\frac{\pi}{2} , \quad (39)$$

$$C_1 = \frac{\sqrt{27}}{2} - \frac{\sqrt{27}}{2} \sin \phi - \sqrt{2} \cos \phi \approx 3 , \quad (40)$$

$$C_2 = \frac{1}{C_1 + \sqrt{2} \cos \phi} \approx \frac{1}{6} . \quad (41)$$

• *An Observable*

To connect with physical measurements, consider the excess energy $\Delta E(t, k)$ at time t of a mode with wavenumber k . It is given by equation (21) or, equivalently, by:

$$\Delta E(t, k) = \frac{a^3 \dot{M}^2}{8M} + [2k a^2 M - 1]^2 , \quad (42)$$

where we have used (27, 22). We shall be interested in any excess energy ΔE acquired during time evolution through the oscillating regime. The resulting excess energy density $\Delta \rho$ is:

$$\Delta \rho(t, k) = \int \frac{d^3 k}{[2\pi a(t)]^3} \Delta E(t, k) = \frac{1}{2\pi^2 a^3(t)} \int dk k^2 \Delta E(t, k) . \quad (43)$$

Perhaps of more relevance for gravity wave detectors is the amount of gravitational waves energy density $\Delta \rho$ per wavenumber k , and divided by the critical density ρ_{cr} :

$$\frac{d}{d \ln k} \Omega_{\text{gw}}(t, k) \equiv \frac{1}{\rho_{\text{cr}}} \frac{d}{d \ln k} \Delta \rho(t, k) \quad (44)$$

$$= \frac{4G k^3}{3\pi H_{\text{now}}^2 a^3(t)} \Delta E(t, k) . \quad (45)$$

6 The Results

We first define dimensionless variables:

$$\tau \equiv \omega \Delta t , \quad \kappa \equiv \frac{k}{\omega a_{\text{cr}}} , \quad \alpha \equiv \frac{a}{a_{\text{cr}}} , \quad \mathcal{H} \equiv \frac{H}{\omega} , \quad \mathcal{M} \equiv 2k a^2 M \quad (46)$$

and re-express in terms of them the evolution equation (28):

$$\frac{d^2 \mathcal{M}}{d\tau^2} + \mathcal{H} \frac{d\mathcal{M}}{d\tau} + \mathcal{H}^2 (4 - 2\epsilon) \mathcal{M} + \frac{2\kappa^2}{\alpha^2} \left[\frac{1}{\mathcal{M}} - \mathcal{M} \right] = \frac{1}{2\mathcal{M}} \left(\frac{d\mathcal{M}}{d\tau} \right)^2 \quad (47)$$

the asymptotic series expansion (29):

$$\begin{aligned} \mathcal{M}(t, k) = & 1 + \left(1 - \frac{\epsilon}{2}\right) \left(\frac{\mathcal{H}\alpha}{\kappa}\right)^2 \\ & + \left[\frac{9}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{3}{4}\epsilon^3 + \left(\frac{7}{4} - \frac{3\epsilon}{4}\right) \frac{\dot{\epsilon}}{H} + \frac{\ddot{\epsilon}}{8H^2} \right] \left(\frac{\mathcal{H}\alpha}{\kappa}\right)^4 + \dots, \end{aligned} \quad (48)$$

as well as the excess energy (42):

$$\Delta E = \omega \left\{ \frac{\alpha \mathcal{H}^2}{4\kappa \mathcal{M}} \left(\mathcal{M} - \frac{1}{2\mathcal{H}} \frac{\partial \mathcal{M}}{\partial \tau} \right)^2 + \frac{\kappa}{4\alpha \mathcal{M}} [\mathcal{M} - 1]^2 \right\}, \quad (49)$$

and the observable (45):

$$\begin{aligned} \frac{d}{d \ln k} \Omega_{\text{gw}} = & G\omega^2 \times \frac{4}{3\pi} \frac{1}{\mathcal{H}^2} \left(\frac{\kappa}{\alpha}\right)^4 \\ & \times \frac{1}{4\mathcal{M}} \left\{ \left(\frac{\mathcal{H}\alpha}{\kappa}\right)^2 \left[\mathcal{M} - \frac{1}{2\mathcal{H}} \frac{\partial \mathcal{M}}{\partial \tau} \right]^2 + [\mathcal{M} - 1]^2 \right\}. \end{aligned} \quad (50)$$

We then discretize the time interval:

$$\tau \rightarrow \tau_i \equiv i \Delta\tau, \quad (51)$$

and the resulting discretized evolution equation (47) determines \mathcal{M}_{i+2} in terms of \mathcal{M}_{i+1} and \mathcal{M}_i :

$$\begin{aligned} \mathcal{M}_{i+2} = & 2\mathcal{M}_{i+1} - \mathcal{M}_i - \mathcal{H}_i \Delta\tau (\mathcal{M}_{i+1} - \mathcal{M}_i) + \mathcal{H}^2 \Delta\tau^2 (4 - 2\epsilon) \mathcal{M}_i \\ & + \frac{(\mathcal{M}_{i+1} - \mathcal{M}_i)^2}{2\mathcal{M}_i} + \frac{2\kappa^2 \Delta\tau^2}{\alpha_i^2} \left(\frac{1}{\mathcal{M}_i} - \mathcal{M}_i \right). \end{aligned} \quad (52)$$

The results of the combined evolution, using the asymptotic series (48) until either 2 e-foldings before first horizon crossing or criticality⁹ – whichever comes first – and numerical integration of (52) thereafter, are presented in

⁹For *super-horizon* or *near/sub-horizon* modes, respectively.

Figures 1-14. It is important to note the following:

- The conditions used for initializing the numerical integration at 2 e-foldings before first horizon crossing are provided by evaluating (48) at this point.
- The conditions used for initializing the numerical integration at criticality are provided by matching the inflationary solution (31-34) with the oscillatory solution(35-38) at this point, so that the three parameters (ϕ, C_1, C_2) take on the values (39-41).
- The dimensionless wavenumber κ_{cr} which underwent first horizon crossing at $t = t_{\text{cr}}$ and is the wavenumber differentiating *super-horizon* from *sub-horizon* modes equals:

$$\kappa_{\text{cr}} = \mathcal{H}_{\text{cr}} \alpha_{\text{cr}} = \frac{2}{\sqrt{27}} \approx 0.38 . \quad (53)$$

In creating Figures 1-11 we have chosen 136 values of wavenumbers κ ranging from 0.05 (*super-horizon*) to 10 (*sub-horizon*).

- Inspection of Figure 14 makes evident the wild time dependence of the observable $\mathcal{H}^2 \times (\frac{d}{d \ln k} \Omega_{\text{gw}})$. Note that had we not multiplied the observable by \mathcal{H}^2 even wilder variations would occur when \mathcal{H} passes through zero. Hence it is not clear to identify where the transition from the oscillatory to the radiation domination era took place. A reasonable *assumption* is that the transition occurred when $\mathcal{H} > 0$ and $\epsilon = 2$. This determines the corresponding time τ to equal:

$$\text{radiation} \implies \mathcal{H} > 0 \ \& \ \epsilon = 2 \implies \tau \approx 2\pi N_{\text{osc}} . \quad (54)$$

- To study the effect of the duration of the oscillatory regime on the enhancement, we have displayed the results for N_{osc} number of oscillation periods within the regime. The values of N_{osc} used range from $N_{\text{osc}} = 0$ – which corresponds to the usual transition from inflation to radiation domination – to $N_{\text{osc}} = 10$. Figures 1-11 present the results – for $\mathcal{H} > 0$ and $\epsilon = 2$ – in ascending order of N_{osc} values.
- The dimensionless factor $G\omega^2$ is *not* included in the evaluation of $(\frac{d}{d \ln k} \Omega_{\text{gw}})$ as seen in Figures 1-11, 14.

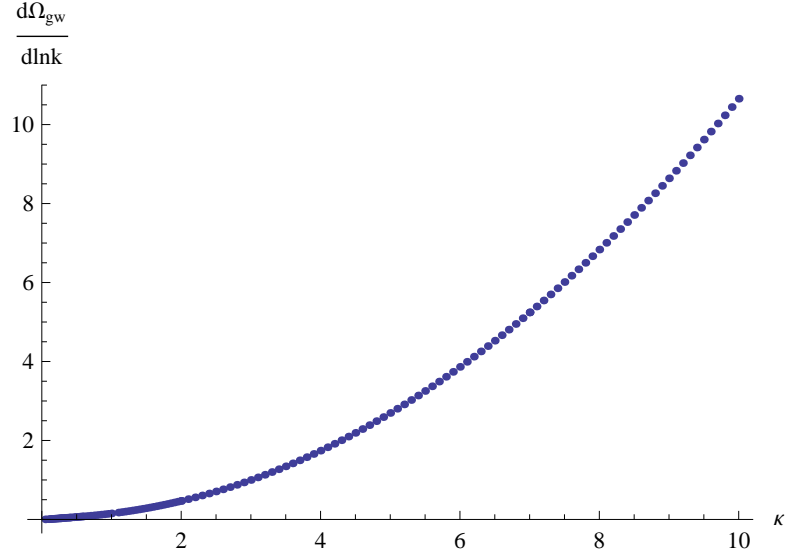


Figure 1: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, for zero periods of oscillations.

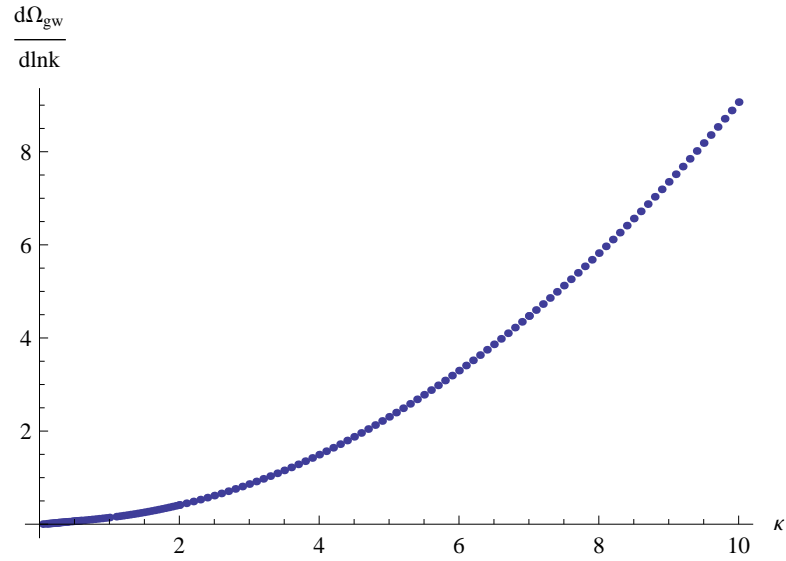


Figure 2: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the first oscillation.

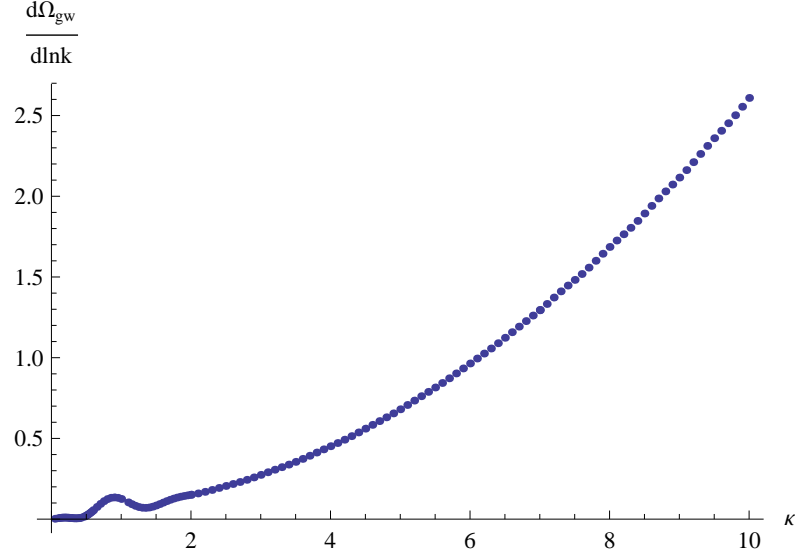


Figure 3: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the second oscillation.

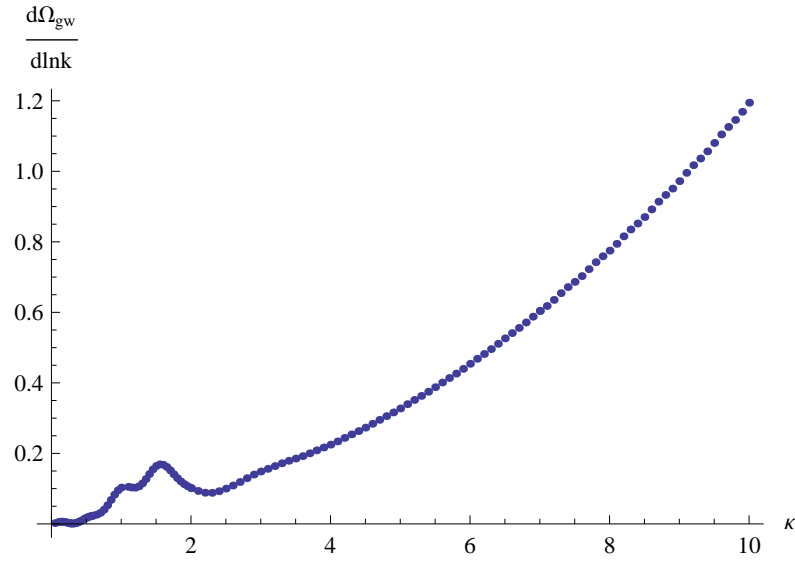


Figure 4: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the third oscillation.

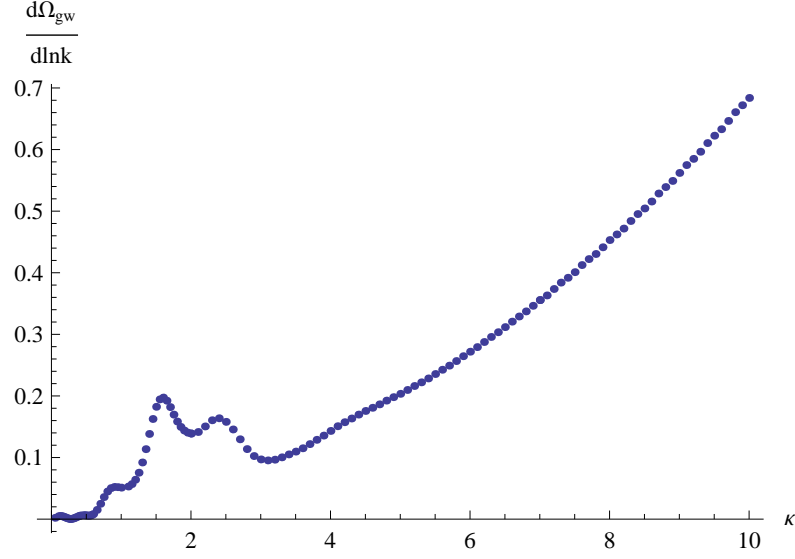


Figure 5: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the fourth oscillation.

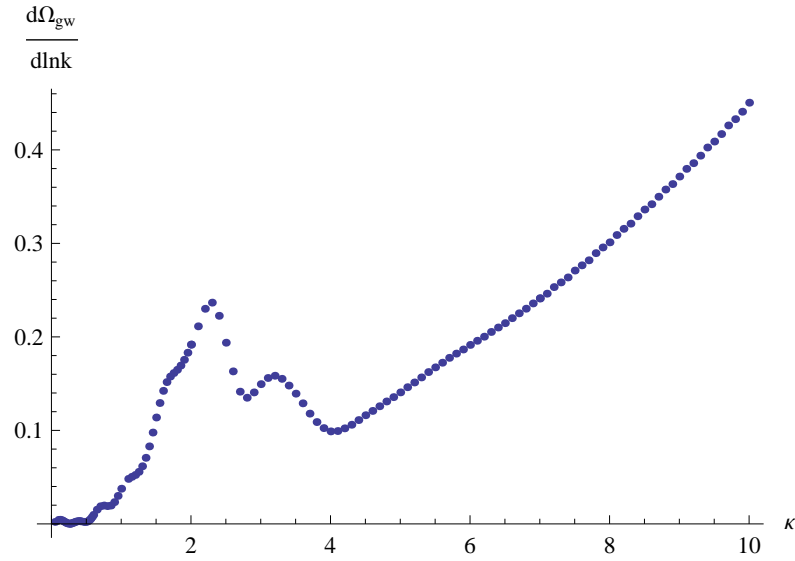


Figure 6: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginnng of the fifth oscillation.

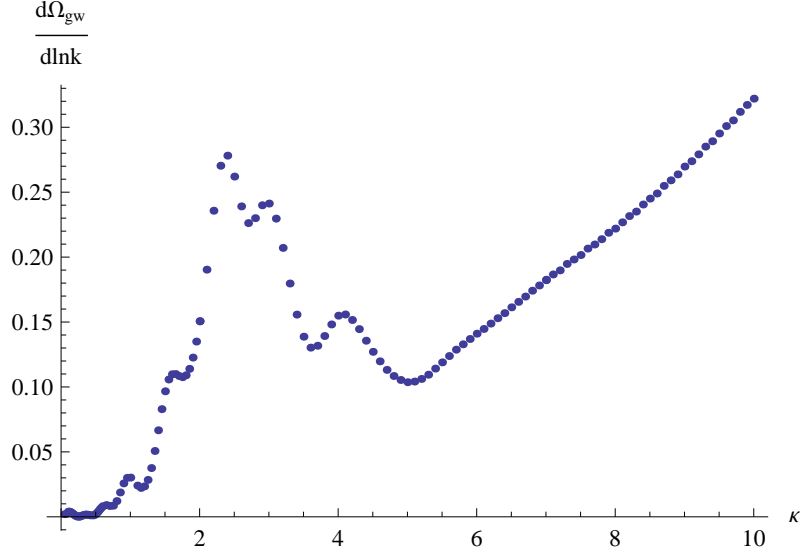


Figure 7: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the sixth oscillation.

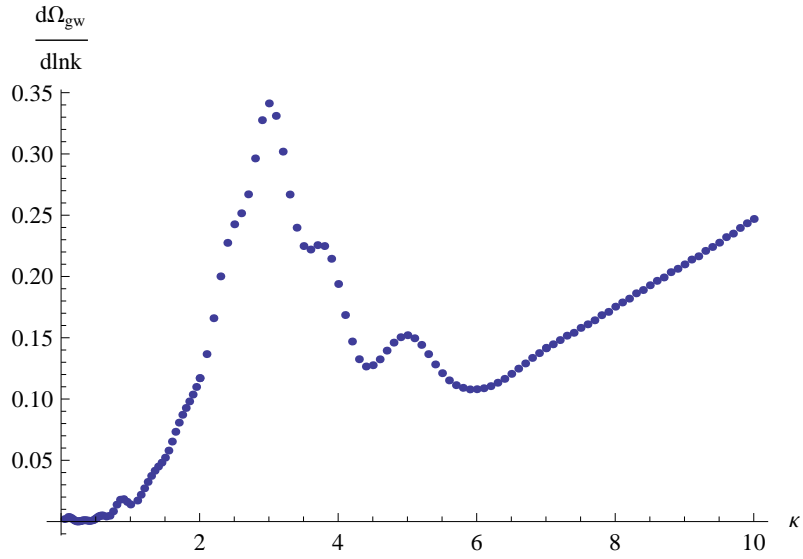


Figure 8: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the seventh oscillation.

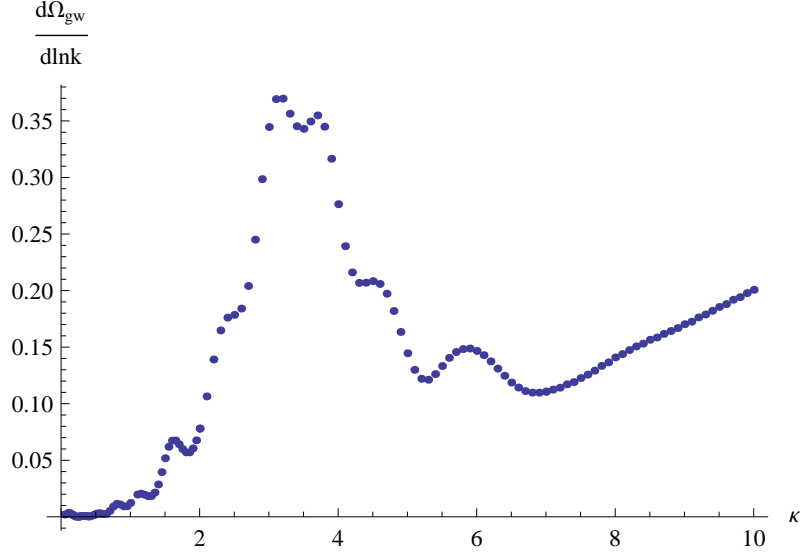


Figure 9: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at the beginning of the eighth oscillation.

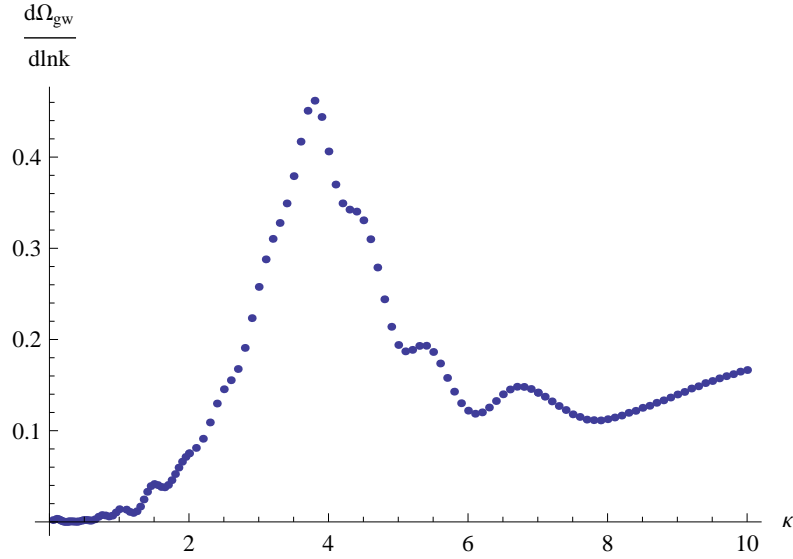


Figure 10: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at beginning of ninth oscillation.

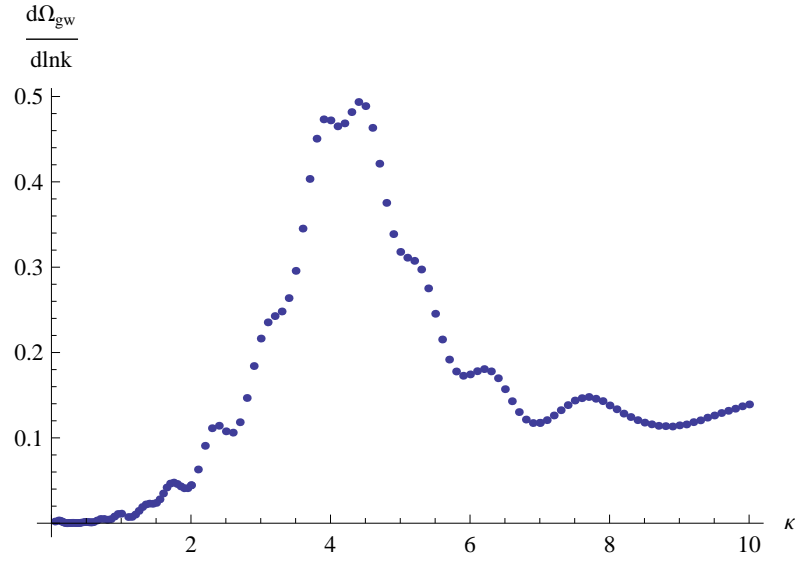


Figure 11: Fraction of the energy density per wavenumber divided by the critical density in gravity waves from our signal, at beginning of tenth oscillation.

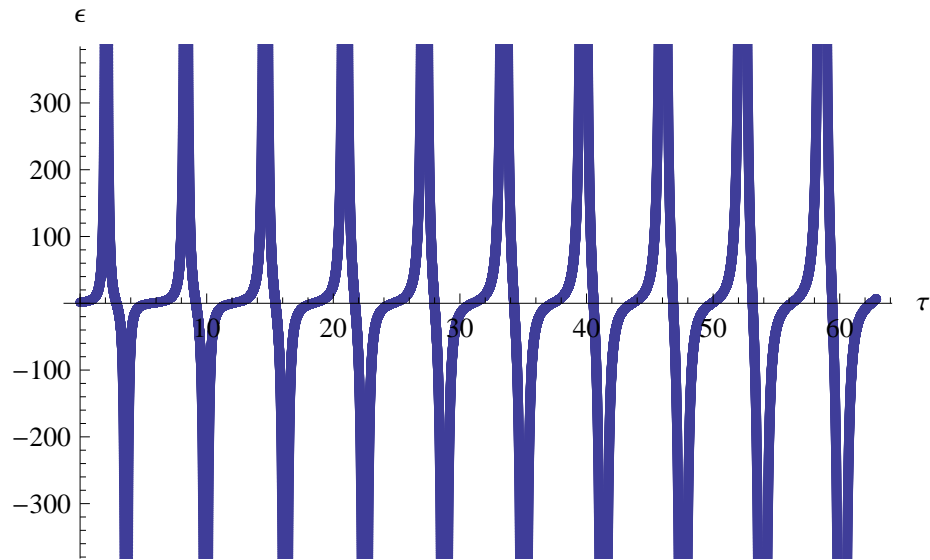


Figure 12: Time evolution of parameter $\epsilon \equiv -\dot{H} H^{-2}$ during the oscillatory regime.

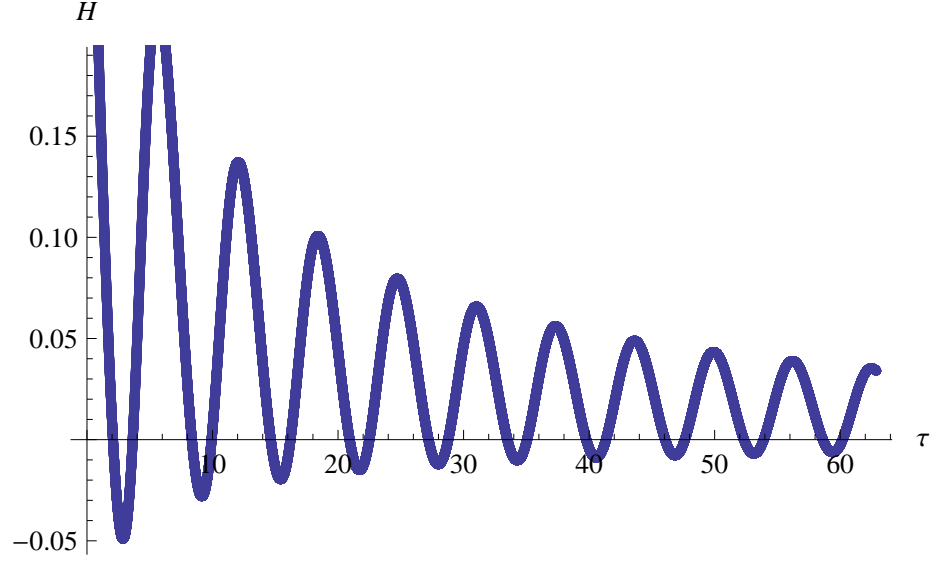


Figure 13: Time evolution of the Hubble parameter \mathcal{H} during the oscillatory regime.

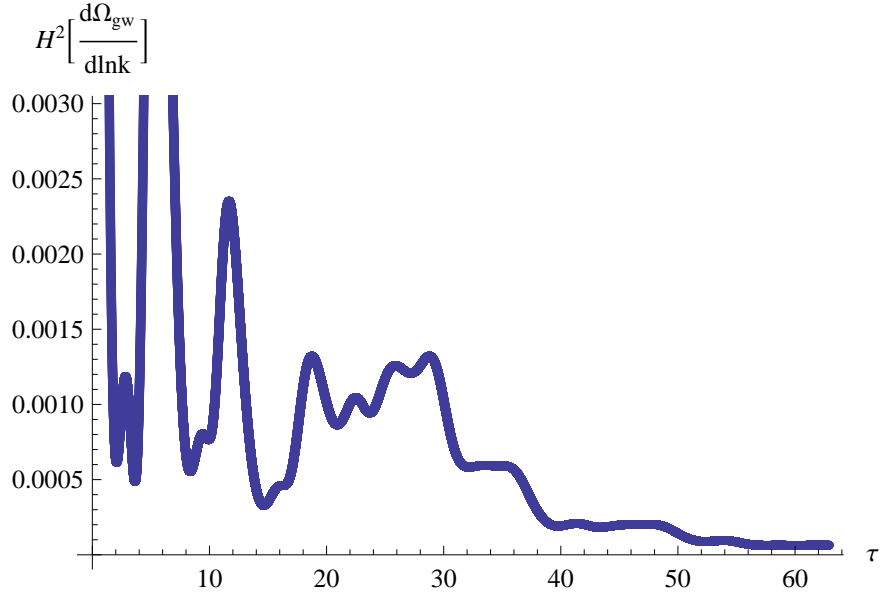


Figure 14: Time evolution of $\mathcal{H}^2 \times (\frac{d}{d\ln k} \Omega_{\text{gw}})$ for $\kappa = 2$ during the oscillatory regime.

7 Physical Consequences

The results of the previous Section allow us to make the following remarks:

- The *existence of the enhancement effect* is confirmed by our analysis. In Section 4 we argued, on physical grounds, that the effect is associated with sign changes of the Hubble parameter. This is explicitly seen in Figures 13-14 where there is a synchronization among the strongest enhancement peaks of the observable (Fig. 14) and sign changes of the Hubble parameter (Fig. 13). Moreover, we see that the effect diminishes with time.
- The *far super-horizon* modes left the horizon many e-foldings before criticality, their mode functions are “frozen” thereafter, and these modes are not affected much from the presence of the oscillating regime.
- The *near-horizon* modes show the enhancement due to resonances close to the oscillatory era frequency ω . Notice that as N_{osc} increases the peak enhancement magnitude increases and shifts towards higher values of κ . Thus, it is the *near/sub-horizon* modes that feel the biggest enhancement.
- The *far sub-horizon* modes show an ever-increasing “tail” with increasing κ (Figs. 1-11). This enhancement is there and should be observed if very high frequency gravity waves become detectable in the future. It is present even when $N_{\text{osc}} = 0$ (Fig. 1) and, hence, it has nothing to do with the existence or not of the oscillatory epoch. Furthermore, we can understand it without resorting to numerical results. When $\kappa \gg 1$ the variable (48) \mathcal{M} becomes essentially unity and the observable (50) simplifies considerably:

$$\left(\frac{\mathcal{H}\alpha}{\kappa}\right)^2 \ll 1 \implies \mathcal{M} \approx 1 \implies \frac{d}{d \ln k} \Omega_{\text{gw}} \approx \# \times \frac{\kappa^2}{\alpha^2} . \quad (55)$$

Therefore, at any fixed time τ , the value of the observable – being proportional to κ^2 – will follow a parabola as κ increases; this is explicitly seen in Figures 1-11. At any fixed κ , the observable is proportional to α^{-2} and its value decreases accordingly as τ increases; this is seen in Figure 14, albeit for $\kappa = 2$.

- The *high-frequency “tail”* discussed above will inevitably lead to ultraviolet divergences. Ultimately it is the correct ultraviolet theory of quantum gravity that will have to address the issue. Nonetheless, an interesting question to answer – within the framework of ordinary perturbative quantum gravity – is which counterterms would absorb the ultraviolet divergences of our observable.

As a first step, we use (48) to expand (50) in powers of $(\frac{\mathcal{H}\alpha}{\kappa})^2$ until we reach ultraviolet convergence. To make the connection with the available counterterms more direct, we also convert – using (44) – from the ratio Ω_{gw} to the excess energy density $\Delta\rho$. The result is:

$$\frac{d}{d \ln k} \Delta\rho(t, k) = \frac{\omega^4}{8\pi^2} \left\{ \mathcal{H}^2 \left(\frac{\kappa}{\alpha} \right)^2 + \left[\frac{3}{2} \epsilon - \frac{3}{4} \epsilon^2 + \frac{\dot{\epsilon}}{2H} \right] \mathcal{H}^4 + O\left(\frac{\mathcal{H}^6 \alpha^2}{\kappa^2} \right) \right\} \quad (56)$$

To make (56) ultraviolet finite, two subtractions are needed: one to renormalize the κ^2 term and one to renormalize the constant term. As we shall see, the two counterterms which absorb the divergences are, respectively:

$$\Delta\mathcal{L}_1 = g_1 R \sqrt{-g} \quad , \quad \Delta\mathcal{L}_2 = g_2 R^2 \sqrt{-g} \quad . \quad (57)$$

These counterterms induce the following stress-energy tensor contributions:

$$\Delta T_{\mu\nu}^1 = 2g_1 \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \quad , \quad (58)$$

$$\Delta T_{\mu\nu}^2 = 2g_2 \left[2R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 2(g_{\mu\nu} \square - D_\mu D_\nu) \right] R \quad . \quad (59)$$

We are interested in the energy density component of the stress-energy tensor for cosmologically relevant (*FRW*) spacetimes; in that case:

$$FRW \implies \Delta T_{00}^1 = 6 g_1 \omega^2 \mathcal{H}^2 \quad , \quad (60)$$

$$FRW \implies \Delta T_{00}^2 = -144 g_2 \omega^4 \left[\frac{3}{2} \epsilon - \frac{3}{4} \epsilon^2 + \frac{\dot{\epsilon}}{2H} \right] \mathcal{H}^4 \quad , \quad (61)$$

and our assertion is established: ΔT_{00}^1 – given by (60) – absorbs the quadratically diverging order \mathcal{H}^2 term in (56), while ΔT_{00}^2 – given by (61) – absorbs the order \mathcal{H}^4 term in (56) which diverges logarithmically.¹⁰

- The *present value of the enhancement* can be straightforwardly computed:

$$\left(\frac{d}{d \ln k} \Omega_{\text{gw}} \right)_{\text{now}} \approx \left(\frac{a_{\text{matter}}}{a_{\text{now}}} \right) \left(\frac{d}{d \ln k} \Omega_{\text{gw}} \right)_{\text{matter}} \quad (62)$$

$$\approx \left(\frac{a_{\text{matter}}}{a_{\text{now}}} \right) \left(\frac{d}{d \ln k} \Omega_{\text{gw}} \right)_{\text{osc}} G \omega^2 \quad (63)$$

$$\approx 0.3 \times 10^{-3} \left(\frac{d}{d \ln k} \Omega_{\text{gw}} \right)_{\text{osc}} 4 \times 10^{-12} \quad . \quad (64)$$

¹⁰The value of g_2 needed to subtract the order \mathcal{H}^4 term in (56) agrees with that first found in 1974 by ‘t Hooft and Veltman [14].

The value of the observable in the oscillatory regime, for given N_{osc} and κ , can be found in Figures 1-11.¹¹ The remaining factor of about 10^{-15} in (64) makes the enhancement effect very small and presently unobservable. The passage from (63) to (64) is valid because after their entrance to the radiation era the gravitational waves behave like any other kind of radiation, and because during the radiation regime – unlike the matter regime – the product $\mathcal{H}^2\alpha^4$ appearing in (50) is constant.

- The *present frequency* of the enhanced waves is given by:

$$f_{\text{now}} = \frac{k}{2\pi a_{\text{now}}} = \frac{\omega\kappa}{2\pi} \left(\frac{a_{\text{cr}}}{a_{\text{now}}} \right) \lesssim 10^9 \text{Hz} \times e^{-\frac{1}{2}\Delta N} \lesssim 10^9 \text{Hz} \quad . \quad (65)$$

For the estimate (65) we used the *near-horizon* value $\kappa \approx 1$ as well as [11]:

$$\left(\frac{a_{\text{cr}}}{a_{\text{now}}} \right) \lesssim e^{-63-\frac{1}{2}\Delta N} \approx 10^{-28} \times e^{-\frac{1}{2}\Delta N} \quad , \quad (66)$$

$$\omega \lesssim 10^{55} H_{\text{now}} \approx 3.2 \times 10^{37} \text{Hz} \quad , \quad (67)$$

where ΔN is the number of oscillatory e-foldings which we expect to be small.

8 Epilogue

From Figure 14 it is evident that the signal is peaked at a narrow band of very high frequencies and is negligible at significantly different frequencies. It would be challenging to detect gravitational radiation at such high frequencies but detectors in that range have been proposed [15]. As noted in the text, the phase of oscillations does not affect modes which experienced first horizon crossing more than a few e-foldings before the end of inflation. The wavelength of our effect is $\lambda = f^{-1} \gtrsim 0.3m$, whereas the smallest scale feature which is currently observed in the cosmic microwave radiation is about $10^{22}m$ [15]! Our model does not change either how matter couples to gravity or the propagation of linearized gravitons, so it has no effect on the spin-down rate of the binary pulsars. The gravity waves we predict will certainly distort how pulsar light propagates, but the short wavelength again seems to preclude a detectable effect. LIGO is not sensitive above frequencies of 7000Hz , which is far too low. The situation is even worse with LISA's high frequency cutoff of 0.1Hz [15].

¹¹As mentioned in Section 6, the displays of the observable $\left(\frac{d}{d \ln k} \Omega_{\text{gw}} \right)_{\text{osc}}$ in all Figures therein lack an overall factor of $G\omega^2$.

Acknowledgements

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